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RANDOM VIBRATION OF LINEAR STRUCTURES

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Abstract—This paper presents a unified method of structural dynamic analysis that is readily applicable to the nonstationary random response analysis of stable linear trusses and frames of two or three dimensional configuration. The method is general enough to treat the structure either as a continuous (distributed) mass system or as a discrete (lumped) mass system, with any form of linear viscous damping. Forcing functions, random in time and/or in space, can be applied anywhere when a frame is considered. However, when a truss is considered, the forces are applied only at the joints.

In evaluating the frequency response function matrix or the impulse response function matrix, the linear graph theory and the transfer matrix technique are employed throughout the formulation so that the configuration of structures are taken into consideration in a most general fashion, permitting a convenient use of a high speed digital computer for numerical computation. The present formulation includes the static structural analysis as a special case.

A number of numerical examples are worked out and the dynamic characteristics of continuous mass systems are compared with those of corresponding discrete mass systems.

1. INTRODUCTION

IN RECENT years, considerable effort has been made in the general area of random vibration as to how the load to mechanical and civil engineering structures can be described as a stochastic process, and how the random load thus idealized as a stochastic process is related to the structural response. Typical examples are the studies of dynamic response characteristics of suspension bridges and buildings subjected to the load such as a gusty wind or an earthquake acceleration.

In these studies, the mean value and the covariance (or correlation) function are two quantities of vital importance in the statistical representation of the excitation and the response process, although they do not necessarily describe the random process completely. In particular, these two functions of the response process play an essential role in the safety analysis of structures subjected to random loading, in estimating fatigue life, in evaluating the probability of catastrophic failure, etc. [1–4]. Furthermore, if the process is Gaussian, these two functions determine the probability density function of any order. Therefore, how to evaluate the mean value and the covariance function of the response process of a general linear structure with the knowledge of the mean value and the covariance function of the excitation process or the equivalent, is the major concern of the present study.

It is assumed that the random processes considered in the present paper possess the properties of continuity, differentiability and integrability at least in the sense of mean square [2, 4].

Let the response at point j of the structure be denoted by $Y_j(t)(j = 1, 2, \ldots, m)$ and the random excitation at point k denoted by $P_k(t)(k = 1, 2, \ldots, n)$. Let $h_{jk}(t)$ be the impulse response function at point j due to the impulse $\delta(t)$ applied at point k. Define the frequency response function $H_{jk}(\omega)$ so that the response at point j to the input $e^{i\omega t}$ at point k is $H_{jk}(\omega)\,e^{i\omega t}$, where i denotes the imaginary unit. Then, the excitation response relationship is given by

$$Y(t) = \int_{-\infty}^{\infty} h(t-\tau)P(\tau)\,d\tau \tag{1}$$

where the excitation $P(t)$ and the response $Y(t)$ are column vectors with $P_k(t)$ as the kth element and $Y_j(t)$ as the jth element, respectively. The impulse response function and the frequency response function are related through the Fourier transform pair;

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt \tag{2}$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\,e^{i\omega t}\,d\omega \tag{3}$$

where the frequency response function matrix $H(\omega)$ is a $m \times n$ matrix with $H_{jk}(\omega)$ as the j–k element and the impulse response function matrix $h(t)$ is a $m \times n$ matrix with $h_{jk}(t)$ as the j–k element.

It can be shown from equations (1) to (3) that the mean value function $m_Y(t)$ of the response is

$$m_Y(t) = E[Y(t)] = \int_{-\infty}^{\infty} h(t-\tau)m_P(\tau)\,d\tau \tag{4}$$

or

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\bar{m}_P(\omega)\,e^{i\omega t}\,d\omega \tag{5}$$

where E denotes the expectation and

$$m_P(t) = E[P(t)] \tag{6}$$

$$\bar{m}_P(t) = E\left[\int_{-\infty}^{\infty} P(t)\,e^{-i\omega t}\,dt\right]. \tag{7}$$

Let $K_{YY}(t_1, t_2)$ denote the covariance function matrix with the covariance function $K_{Y_jY_{j'}}(t_1, t_2)$ of $Y_j(t_1)$ and $Y_{j'}(t_2)$ as the j–j' element $(j, j' = 1, 2, \ldots, m)$;

$$K_{YY}(t_1, t_2) = E[\{Y(t_1)-m_Y(t_1)\}\{Y(t_2)-m_Y(t_2)\}'] \tag{8}$$

where the prime denotes the transpose of a matrix. Then, it can be shown that

$$K_{YY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1-\tau_1)K_{PP}(\tau_1, \tau_2)h'(t_2-\tau_2)\,d\tau_1\,d\tau_2 \tag{9}$$

or

$$K_{YY}(t_1, t_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{YY}(\omega_1, \omega_2)\, e^{i(\omega_1 t_1 - \omega_2 t_2)}\, d\omega_1\, d\omega_2 \tag{10}$$

with

$$K_{PP}(t_1, t_2) = E[\{P(t_1) - m_P(t_1)\}\{P(t_2) - m_P(t_2)\}'] \tag{11}$$

$$S_{YY}(\omega_1, \omega_2) = H(\omega_1) S_{PP}(\omega_1, \omega_2) H^{*\prime}(\omega_2) \tag{12}$$

and

$$S_{PP}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{PP}(t_1, t_2)\, e^{-i(\omega_1 t_1 - \omega_2 t_2)}\, dt_1\, dt_2. \tag{13}$$

The mean value and the covariance function of the response can be evaluated either in the time domain using equations (4) and (9) in which case the impulse response function matrix $h(t)$ of the structure, the mean value function vector $m_P(t)$ and the covariance function matrix $K_{PP}(t_1, t_2)$ of the excitation must be known, or in the frequency domain with the aid of equations (5) and (10) where the knowledge of the frequency response function matrix $H(\omega)$, the Fourier transformed mean value function vector $\bar{m}_P(\omega)$ and the generalized spectral density matrix $S_{PP}(\omega_1, \omega_2)$ of the excitation are required.

Since statistical characteristics of the excitation process is assumed to be known in terms of the mean value and covariance function or in terms of their Fourier transforms, it only remains to determine the impulse response function matrix and the frequency response function matrix of the structure in order that the formulation given in equations (1)–(13) can be used for the mean value and the covariance function of the response. The emphasis in the present study, therefore, mainly placed on the techniques of estimating the impulse response function matrix and the frequency response function matrix.

2. PRELIMINARIES

(1) *Structures considered*

Consider a stable frame or truss of arbitrary configuration consisting of straight members and supports with no release. Choose the points of support and of intersection of members as nodes. The nodes are identified either by upper case letters in alphabetical order or by positive integers $1, 2, \ldots, \bar{N}$ with the nodes at supports (datum nodes) last, where $\bar{N}$ is the total number of nodes. Number and orient individual members (branches) arbitrarily. Thus a frame or truss is associated with an oriented linear graph. In Fig. 1, an oriented linear graph is drawn for a frame structure.

It is assumed that the cross-section of each member is uniform for trusses whereas it can be piecewise uniform for frames in which case nodes are created at the points of uniformity change in addition to those at the points of intersections.

(2) *Branch–node and node–node incidence matrix*

Consider branch j that is oriented from node A (the initial node) to node B (the final node). Node $A(B)$ is said to be positively (negatively) incident on branch j and branch j is

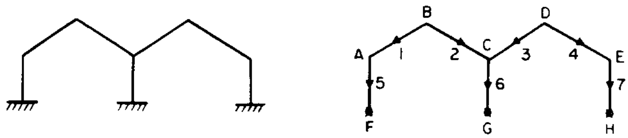

FIG. 1. A frame structure and its associated linear graph.

said to be positively (negatively) incident on node $A(B)$. The initial (final) node of a branch is said to be positively (negatively) incident on the final (initial) node of that branch.

To specify the connectivity of a linear graph, the augmented branch-node incidence matrix A^* [5, 6] is employed. The rows of A^* correspond to the branches and the columns to the nodes, and its j–J element, a_{jJ}, is equal to $+1$, -1 or 0 depending on whether branch j is positively, negatively or not incident on node J. Clearly, the matrix A^* contains all the information of connectivity and orientation of a linear graph.

The matrix A^* with the datum columns (the columns associated with the datum nodes) removed is referred to as the branch–node incidence matrix A.

The node–node incidence matrix E is defined among the non-datum nodes in such a way that its I–J element, e_{IJ}, is equal to $+1$, -1 or 0 depending on whether node J is positively, negatively, or not incident on node I with the provision that $e_{IJ} = 0$ if $I = J$. Evidently, the matrix E is derivable from the matrix A.

(3) *Random excitations*

When a truss is considered, the source of the random excitation is limited to a set of concentrated random forces acting at the nodes only. The random displacement and acceleration excitations at nodes can also be considered. For a frame, however, the concentrated random forces (including those in the form of couples) as well as the random displacement and acceleration excitations can be applied at any point of the structure. When these excitations are applied to the frame at those points other than the intersections of the members, additional nodes have to be created at the points of application of such excitations. Also wherever the lumped masses are attached to the structure other than at the intersection of the members, additional nodes have to be created at these points. The distributed random excitation should be approximated by the concentrated random excitation [2, p. 175].

(4) *Coordinate systems*

Let a global coordinate system fixed in space be denoted by the rectangular right-hand axes (ξ, η, ζ) with an arbitrary orientation. Associated with each branch, say branch j, construct a local rectangular right-hand coordinate system (x_j, y_j, z_j). This system is fixed with respect to the global coordinate system such that, in the undeformed state, x_j coincides with the axis of the member (branch) while y_j and z_j coincide with two principal axes of the initial cross section of the branch. Let Λ_j be the (orthogonal) transformation matrix between the coordinate systems (x_j, y_j, z_j) and (ξ, η, ζ).

(5) *Branch and node quantities*

Unless otherwise stated, branch quantities (branch forces and branch displacements) have a superscript G if their components are with respect to the global coordinate system, whereas the nodal quantities are always referred to the global coordinate system without superscript.

Let the resultant forces and moments acting on initial end I and final end F of branch j of a frame (the end branch forces) be represented by (6×1) vectors with components in the local coordinate system,

$$ {}_IT_j = [{}_IT_{j1}\ {}_IT_{j2}\ {}_IT_{j3}\ {}_IT_{j4}\ {}_IT_{j5}\ {}_IT_{j6}]' $$

$$ {}_FT_j = [{}_FT_{j1}\ {}_FT_{j2}\ {}_FT_{j3}\ {}_FT_{j4}\ {}_FT_{j5}\ {}_FT_{j6}]' $$

in which ${}_IT_{j1}$, ${}_IT_{j2}$ and ${}_IT_{j3}$ are, respectively, the x_j, y_j and z_j components of the force acting on the initial end of the branch whereas ${}_IT_{j4}$, ${}_IT_{j5}$ and ${}_IT_{j6}$ are, respectively, the x_j, y_j, and z_j components of the couple on the initial end. Similar definitions apply to ${}_FT_{jk}(k = 1, \ldots 6)$ at the final end.

Similarly, the local components of the displacements (including rotations) at the initial and the final end are the end branch displacements ${}_IU_j$ and ${}_FU_j$ where

$$ {}_IU_j = [{}_IU_{j1}\ {}_IU_{j2}\ {}_IU_{j3}\ {}_IU_{j4}\ {}_IU_{j5}\ {}_IU_{j6}]' $$

$$ {}_FU_j = [{}_FU_{j1}\ {}_FU_{j2}\ {}_FU_{j3}\ {}_FU_{j4}\ {}_FU_{j5}\ {}_FU_{j6}]' $$

These quantities can be expressed in the global coordinate system by the following transformation

$$ {}_IT_j^G = R_j'\,{}_IT_j, \qquad {}_FU_j^G = R_j'\,{}_FU_j $$

in which

$$ R_j = \begin{bmatrix} \Lambda_j & 0 \\ 0 & \Lambda_j \end{bmatrix}. $$

Moreover, let T_j and U_j denote the branch forces and branch displacements, respectively, at any cross section of branch j, i.e. $T_j(x_j = 0) = {}_IT_j$, $T_j(x_j = l_j) = {}_FT_j$, etc. where l_j is the length of the member (branch j).

Because of the assumption of no release, the displacements of the end cross-sections of those members that meet at a common node are identical. For example, if node J is the initial node of the branch j and is the final node of the branch k, then

$$ {}_IU_j^G = {}_FU_k^G. $$

Hence, the nodal displacement at node J is defined by

$$ {}_JU \equiv {}_IU_j^G \equiv {}_FU_k^G. $$

Furthermore, introduce a (6×1) vector ${}_JP$ referred to as the nodal force at node J whose elements are the global components of the external force applied at node J

$$ {}_JP = [{}_JP_1\ {}_JP_2\ {}_JP_3\ {}_JP_4\ {}_JP_5\ {}_JP_6]' $$

in which ${}_JP_1$, ${}_JP_2$ and ${}_JP_3$ are, respectively, the ξ, η and ζ components of the applied force and ${}_JP_4$, ${}_JP_5$ and ${}_JP_6$ are, respectively, the ξ, η and ζ components of the applied couple.

A $(6B \times 1)$ vector T and $(6N \times 1)$ vectors U and P are then defined as follows:

$$T = [{}_I T_1 \; {}_I T_2 \ldots {}_I T_j \ldots {}_I T_B]'$$
$$U = [{}_1 U \; {}_2 U \ldots {}_J U \ldots {}_N U]'$$
$$P = [{}_1 P \; {}_2 P \ldots {}_J P \ldots {}_N P]'$$

where B is the total number of branches and N is the total number of non-datum nodes in the structure.

The quantities defined above can be employed for truss problems if the last three components of ${}_I T_j$, ${}_F T_j$, ${}_I U_j$, ${}_F U_j$, ${}_J U$ and ${}_J P$ are dropped so that now these are all (3×1) vectors, since they are either identically zero or will be eliminated from the formulation. Also, R_j is to be replaced by Λ_j for the coordinate transformation.

(6) *Sign convention*

The standard right-hand rule is adopted as sign convention for the quantities discussed above. Hence, the components of the displacement and of the nodal force are positive if they are in the same direction as the corresponding reference coordinates (local or global). The components of the resultant force (concentrated force and couple), acting on a cross-section of branch j with positive outward normal in reference to the direction of x_j-axis, are positive if they are in the same direction as the corresponding local coordinates, while the components of the resultant force on a cross-section with negative outward normal, are positive if they are in the negative direction of the corresponding local coordinates. This convention is illustrated in Fig. 2 for ${}_I T_i$.

(7) *Steady state vibration*

To determine the frequency response function matrix, the steady state vibration of the quantities described in the preceding discussion are considered in the complex form with

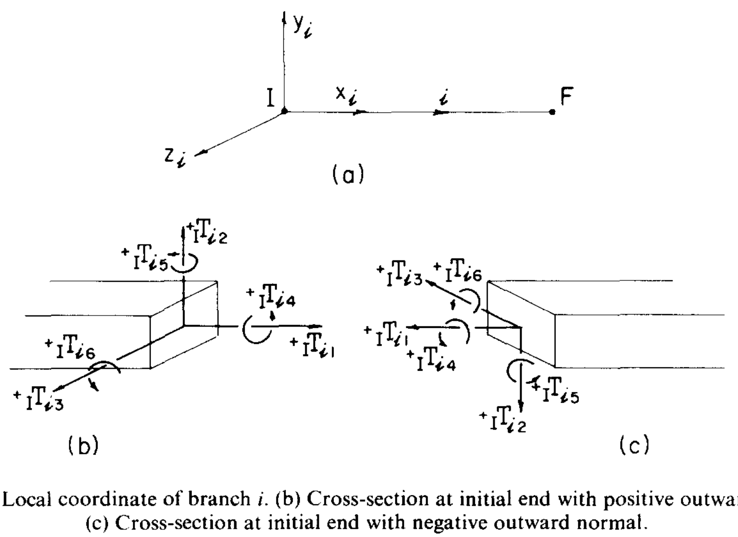

FIG. 2. (a) Local coordinate of branch i. (b) Cross-section at initial end with positive outward normal. (c) Cross-section at initial end with negative outward normal.

corresponding lower case letters indicating the complex amplitude;

$$\left.\begin{aligned} {}_I T_j &= {}_I\tau_j\, e^{i\omega t} = [{}_I\tau_{j1}\; {}_I\tau_{j2} \ldots {}_I\tau_{j6}]'\, e^{i\omega t} \\ T_j(x) &= \tau_j(x)\, e^{i\omega t} = [\tau_{j1}(x)\tau_{j2}(x) \ldots \tau_{j6}(x)]'\, e^{i\omega t} \end{aligned}\right\} \tag{14a}$$

etc.

$$\left.\begin{aligned} T &= \tau\, e^{i\omega t} = [{}_I\tau_1\; {}_I\tau_2 \ldots {}_I\tau_B]'\, e^{i\omega t} \\ U &= u\, e^{i\omega t} = [{}_1u\; {}_2u \ldots {}_Nu]'\, e^{i\omega t} \\ P &= \mathrm{p}\, e^{i\omega t} = [{}_1p\; {}_2p \ldots {}_Np]'\, e^{i\omega t} \end{aligned}\right\}. \tag{14b}$$

3. FREQUENCY RESPONSE MATRIX

(1) *Frames*

(a) *Branch vibration.* Consider the vibration of branch j with the following notations; A_j = cross sectional area, E_j = Young's modulus, $c_{j1} = c_{j2} = c_{j3}$ = retardation time of normal stress–strain relation, m_j = mass per unit length, ${}_jc_1$, ${}_jc_2$, ${}_jc_3$ and ${}_jc_4$ = coefficients of linear viscous (external) damping associated, respectively, with the extensional vibration, flexural vibrations in the x_j-y_j plane and in the x_j-z_j plane and torsional vibration, I_{jz} and I_{jy} = moments of inertia of the cross section about z_j and y_j axes, G_j = shear modulus, J_j = Saint Venant constant of uniform torsion, r_j = radius of gyration and l_j = length.

First, consider the flexural vibration in x_j-y_j plane. The equation of motion and the relation between the branch force and the branch displacement can be written as

$$E_jI_{jz}\frac{\partial^4}{\partial x_j^4}(U_{j2}+c_{j2}\dot{U}_{j2})+m_j\ddot{U}_{j2}+{}_jc_2\dot{U}_{j2}=0 \tag{15}$$

$$T_{j2} = -\frac{\partial T_{j6}}{\partial x_j} \tag{16}$$

$$T_{j6} = E_jI_{jz}\frac{\partial^2}{\partial x_j^2}(U_{j2}+c_{j2}\dot{U}_{j2}) \tag{17}$$

where dots indicate the differentiation with respect to time.

The steady state solution to equation (15) in the form of equation (14), is obtained as

$$\begin{aligned} u_{j2}(x_j) = {} & A_1\sin(\lambda_{j2}x_j/l_j)+A_2\cos(\lambda_{j2}x_j/l_j) \\ & +A_3\sinh(\lambda_{j2}x_j/l_j)+A_4\cosh(\lambda_{j2}x_j/l_j) \end{aligned} \tag{18}$$

where

$$\lambda_{j2}^4 = (m_j\omega^2 - i\omega\,{}_jc_2)l_j^4/E_jI_{jz}(1+i\omega c_{j2}) \tag{19}$$

and A_1, A_2, A_3 and A_4 are to be determined from the following boundary conditions.

$${}_Iu_{j2} = u_{j2}(0) \tag{20}$$

$${}_Iu_{j6} = u'_{j2}(0) \tag{21}$$

$${}_Fu_{j2} = u_{j2}(l_j) \tag{22}$$

$${}_Fu_{j6} = u'_{j2}(l_j) \tag{23}$$

$$ {}_I\tau_{j2} = \tau_{j2}(0) = -E_j I_{jz}(1+i\omega c_{j2})u'''_{j2}(0) \tag{24}$$

$$ {}_I\tau_{j6} = \tau_{j6}(0) = E_j I_{jz}(1+i\omega c_{j2})u''_{j2}(0) \tag{25}$$

$$ {}_F\tau_{j2} = \tau_{j2}(l_j) = -E_j I_{jz}(1+i\omega c_{j2})u'''_{j2}(l_j) \tag{26}$$

$$ {}_F\tau_{j6} = \tau_{j6}(l_j) = E_j I_{jz}(1+i\omega c_{j2})u''_{j2}(l_j) \tag{27}$$

where primes denote the differentiation with respect to x_j. Hence, if the solution in equation (18) satisfying equations (20), (21), (24) and (25) as boundary conditions, is substituted into the last two equations, then, the transfer equation is obtained as

$$\begin{bmatrix} {}_F\tau_{j2} \\ {}_F\tau_{j6} \end{bmatrix} = \begin{bmatrix} {}_jf_{11} & {}_jf_{12} \\ {}_jf_{21} & {}_jf_{22} \end{bmatrix} \begin{bmatrix} {}_I\tau_{j2} \\ {}_I\tau_{j6} \end{bmatrix} + \begin{bmatrix} {}_j\bar{f}_{11} & {}_j\bar{f}_{12} \\ {}_j\bar{f}_{21} & {}_j\bar{f}_{22} \end{bmatrix} \begin{bmatrix} {}_Iu_{j2} \\ {}_Iu_{j6} \end{bmatrix} \qquad (28)^*$$

Furthermore, if the solution in equation (18) that satisfies equations (20) to (23) as boundary conditions, is employed in equations (24) and (25), then the end branch force–displacement equation is obtained as

$$\begin{bmatrix} {}_I\tau_{j2} \\ {}_I\tau_{j6} \end{bmatrix} = \begin{bmatrix} K_{j2} & 0 \\ 0 & K_{j6} \end{bmatrix} \begin{bmatrix} {}_jb_{11} & {}_jb_{12} & {}_jb_{13} & {}_jb_{14} \\ {}_jb_{21} & {}_jb_{22} & {}_jb_{23} & {}_jb_{24} \end{bmatrix} \begin{bmatrix} {}_Iu_{j2} \\ {}_Iu_{j6} \\ {}_Fu_{j2} \\ {}_Fu_{j6} \end{bmatrix} \qquad (29)^*$$

The formulation can be made to include the effect of a constant axial force, for example, due to a static loading, on the flexural vibration with a slight modification [7].

In a similar fashion, the transfer equations and the end branch force–displacement equations for the extensional vibration [equations (30a) and (31a)] the flexural vibration in the x_j–z_j plane [equations (30b) and (31b)] and the torsional vibration [equations (30c) and (31c)] are obtained as follows:

$$ {}_F\tau_{j1} = {}_I\tau_{j1}\cos\lambda_{j1} - (m_j\omega^2 - i\omega\, {}_jc_1)\frac{\sin\lambda_{j1}}{\lambda_{j1}} l_j\, {}_Iu_{j1} \qquad (30a)^*$$

$$\begin{bmatrix} {}_F\tau_{j3} \\ {}_F\tau_{j5} \end{bmatrix} = \begin{bmatrix} {}_jg_{11} & {}_jg_{12} \\ {}_jg_{21} & {}_jg_{22} \end{bmatrix} \begin{bmatrix} {}_I\tau_{j3} \\ {}_I\tau_{j5} \end{bmatrix} + \begin{bmatrix} {}_j\bar{g}_{11} & {}_j\bar{g}_{12} \\ {}_j\bar{g}_{21} & {}_j\bar{g}_{22} \end{bmatrix} \begin{bmatrix} {}_Iu_{j3} \\ {}_Iu_{j5} \end{bmatrix} \qquad (30b)^*$$

$$ {}_F\tau_{j4} = {}_I\tau_{j4}\cos\lambda_{j4} - (m_j r_j^2\omega^2 - i\omega\, {}_jc_4)\frac{\sin\lambda_{j4}}{\lambda_{j4}} l_j\, {}_Iu_{j4} \qquad (30c)^*$$

$$ {}_I\tau_{j1} = K_{j1}\frac{\lambda_{j1}}{\sin\lambda_{j1}}({}_Fu_{j1} - {}_Iu_{j1}\cos\lambda_{j1}) \qquad (31a)^*$$

$$\begin{bmatrix} {}_I\tau_{j3} \\ {}_I\tau_{j5} \end{bmatrix} = \begin{bmatrix} K_{j3} & 0 \\ 0 & K_{j5} \end{bmatrix} \begin{bmatrix} {}_j\bar{b}_{11} & {}_j\bar{b}_{12} & {}_j\bar{b}_{13} & {}_j\bar{b}_{14} \\ {}_j\bar{b}_{21} & {}_j\bar{b}_{22} & {}_j\bar{b}_{23} & {}_j\bar{b}_{24} \end{bmatrix} \begin{bmatrix} {}_Iu_{j3} \\ {}_Iu_{j5} \\ {}_Fu_{j3} \\ {}_Fu_{j5} \end{bmatrix} \qquad (31b)^*$$

$$ {}_I\tau_{j4} = K_{j4}\frac{\lambda_{j4}}{\sin\lambda_{j4}}({}_Fu_{j4} - {}_Iu_{j4}\cos\lambda_{j4}) \qquad (31c)^*$$

* In what follows, the definitions of undefined quantities and symbols in the equations with an asterisk are given in Appendix I.

In the present formulation, the effect of warping on the torsional vibration is neglected, although it can be taken into account with a slight modification [7].

The transfer equations given in equations (28) and (30) and the end branch force–displacement relations given in equations (29) and (31) can be written in the following matrix form;

$$ {}_F\tau_j = B_j\,{}_I\tau_j + D_j R_j\,{}_I u_j^G \tag{32*} $$

$$ {}_I\tau_j = K_j(F_j R_j\,{}_I u_j^G + W_j R_j\,{}_F u_j^G) \tag{33*} $$

where the rotational transformations

$$ {}_I u_j = R_j\,{}_I u_j^G, \qquad {}_I u_j^G = R_j'\,{}_I u_j $$

has been used and

$$ K_j = [K_{jk}] \qquad k = 1, 2, \ldots 6 \tag{34} $$

When the jth member is massless, i.e. $m_j = 0$ and therefore $\lambda_{jk} = 0$ assuming ${}_jc_k = 0$ ($k = 1, 2, 3, 4$), one can reduce the transcendental elements of B_j, D_j, F_j and W_j into constants by taking the limit as λ_{jk} ($k = 1, 2, 3, 4$) approach zero. The resulting matrices $\bar{B}_j$, $\bar{D}_j$, $\bar{F}_j$ and $\bar{W}_j$ thus contain no transcendental elements and are not functions of ω (see Appendix for explicit expression of $\bar{B}_j$, $\bar{D}_j$, $\bar{F}_j$ and $\bar{W}_j$).

In the preceding formulation, the complex damping can be introduced if $i\omega c_{jk}(k = 1, \ldots 4)$ are replaced by $i\dot{\alpha}_{jk}(\alpha_{jk} = \text{constant})$. Also, it should be pointed out that any form of linear viscous damping arising from linear viscoelastic stress–strain relations can be considered by replacing $E_j(1 + i\omega c_{jk})$ etc. by $i\omega\bar{G}_{jk}(\omega)$, where $\bar{G}_{jk}(\omega)$ are the Fourier transforms of the appropriate relaxation moduli.

(b) *Nodal vibrations.* The equation of motion at a non-datum node, say at node J, can be written as

$$ \sum_j {}_I T_j^G + \sum_k {}_F T_k^G + {}_J P = {}_J M\,{}_J\ddot{U} + {}_J\bar{c}\,{}_J\dot{U} \tag{35} $$

with the index j refers to those branches positively incident on J while k to those negatively incident on J and

$$ {}_J M = \left\lceil {}_J M_l \right\rfloor, \qquad l = 1, 2, \ldots 6 \tag{36} $$

$$ {}_J\bar{c} = \left\lceil {}_J\bar{c}_l \right\rfloor, \qquad l = 1, 2, \ldots 6 \tag{37} $$

where ${}_J M_1 = {}_J M_2 = {}_J M_3$ indicate the mass at node J and ${}_J M_4$, ${}_J M_5$ and ${}_J M_6$ are the moment of inertia of that mass about the axis passing through its centroid and parallel to the ξ, η and ζ axes respectively, while ${}_J\bar{c}_1$, ${}_J\bar{c}_2$ and ${}_J\bar{c}_3$ (possibly ${}_J\bar{c}_1 = {}_J\bar{c}_2 = {}_J\bar{c}_3$) are coefficients of linear viscous damping associated with translation of node J in the ξ, η and ζ directions and ${}_J\bar{c}_4$, ${}_J\bar{c}_5$ and ${}_J\bar{c}_6$ (possibly ${}_J\bar{c}_4 = {}_J\bar{c}_5 = {}_J\bar{c}_6$) are coefficients of linear viscous damping associated with rotation of the node about the axes passing through the centroid of the mass and parallel to the ξ, η and ζ axes.

For the steady state solution of the form equation (14), equation (35) yields

$$\sum_j {}_I\tau_j^G + \sum_k {}_F\tau_k^G + {}_Jp = (-\omega^2\, {}_JM + i\omega\, {}_J\bar{c})\, {}_Ju \tag{38}$$

which, with the aid of equation (32), can be written as,

$$ {}_Jp + {}_JZ\, {}_Ju - \sum_k R_k' D_k R_k\, {}_Iu_k^G = -\sum_j R_j'\, {}_I\tau_j + \sum_k R_k' B_k\, {}_I\tau_k \tag{39)*}$$

with

$$ {}_JZ = \left[{}_JM_l\omega^2 - i\omega\, {}_J\bar{c}_l\right], \qquad l = 1, 2, 3, \ldots 6. \tag{40}$$

(c) *System vibrations.* Define system matrices $K, Z, \bar{Q}, Q$ and Y, each element of which is a matrix of the individual branch quantities or of the individual nodal quantities.

$$K = \left[K_j\right], \qquad j = 1, 2, \ldots B \tag{41}$$

$$Z = \left[{}_JZ\right], \qquad J = 1, 2, \ldots N \tag{42}$$

$$\bar{Q} = \left[\bar{q}_{jJ}\right], \qquad (j = 1, 2, \ldots B; J = 1, 2, \ldots N) \tag{43}$$

with

$$\bar{q}_{jJ} = \begin{cases} 0 & \text{if } a_{jJ} = 0 \\ F_j R_j & \text{if } a_{jJ} = +1 \\ W_j R_j & \text{if } a_{jJ} = -1 \end{cases} \tag{44)*}$$

$$Q = [q_{jJ}] \qquad (j = 1, 2, \ldots B; J = 1, 2, \ldots N) \tag{45}$$

with

$$q_{jJ} = \begin{cases} 0 & \text{if } a_{jJ} = 0 \\ -R_j & \text{if } a_{jJ} = +1 \\ B_j' R_j & \text{if } a_{jJ} = -1 \end{cases} \tag{46)*}$$

where a_{jJ} is the j–J element of branch–node incidence matrix. B is the total number of branches and N the total number of non-datum nodes. The matrices Q and $\bar{Q}$ are called the modified branch–node incidence matrices.

$$Y = [y_{IJ}] \qquad (I, J = 1, 2, \ldots N) \tag{47}$$

with

$$y_{IJ} = \begin{cases} -R_k' D_k R_k & \text{if } e_{IJ} = 1 \text{ and } k \text{ denotes the branch connecting nodes } I \text{ and } J. \\ 0 & \text{Otherwise} \end{cases} \tag{48)*}$$

where e_{IJ} is the I–J element of the node–node incidence matrix.

With the aid of these system matrices, the branch force–nodal displacement relationship of the system

$$\tau = K\bar{Q}u \tag{49}$$

is derived from equation (33) and the equations of motion at non-datum nodes

$$p + (Z + Y)u = Q'\tau \tag{50}$$

are derived, from equation (39).

Hence, it follows from equations (49) and (50) that

$$u = [Q'K\bar{Q} - (Z + Y)]^{-1}p \tag{51}$$

$$\tau = K\bar{Q}[Q'K\bar{Q} - (Z + Y)]^{-1}p. \tag{52}$$

(d) *Releases.* The preceding discussion is based upon the assumption that no release exists within the frame. When, however, the release occurs at a support node (in the form of a hinge, a roller or an elastic constraint), or at an interior node, a slight modification of the end branch force–displacement equation [equation (33)] can easily be made [7] whereas the transfer equation [equation (32)] remains the same.

(e) *Excitation at supports.* Consider the case where excitations are applied at supports, in the form of random displacement or acceleration, instead of at non-datum nodes. In such a case, the supports which are excited are regarded as non-datum nodes, although they are still designated by upper case letters after the non-support nodes in alphabetical order.

Then, the matrices u, $\bar{Q}$, Q and Y can be partitioned as follows.

$$u = [u_N' \,\vdots\, u_e']', \qquad \bar{Q} = [\bar{Q}_N \,\vdots\, \bar{Q}_e], \qquad Q = [Q_N \,\vdots\, Q_e] \tag{53}$$

$$Y = \left[\begin{array}{c|c} Y_N & Y_e \\ \hline Y_{Ne} & Y_{ee} \end{array}\right] \qquad Z = \left[\begin{array}{c|c} Z_N & 0 \\ \hline 0 & Z_e \end{array}\right] \tag{54}$$

where u_N, $\bar{Q}_N$, Q_N, Z_N and Y_N involved only non-excitation nodes while u_e, $\bar{Q}_e$, Q_e, Z_e, Y_e, Y_{Ne} and Y_{ee} excitation nodes.

Using equations (53) and (54) in equations (49) and (50) with equation (50) containing only the equations of motion for non-excitation nodes, one can solve u_N and τ in terms of the excitations at supports u_e as follows.

$$u_N = [Q_N'K\bar{Q}_N - (Z_N + Y_N)]^{-1}[Y_e - Q_N'K\bar{Q}_e]u_e \tag{55}$$

$$\tau = K\bar{Q}_N[Q_N'K\bar{Q}_N - (Z_N + Y_N)]^{-1}[Y_e - Q_N'K\bar{Q}_e]u_e + K\bar{Q}_e u_e \tag{56}$$

where u_e is to be replaced by $-a_e/\omega^2$ if the excitation is in the form of acceleration.

The modification for the case where excitations, in the form of displacement or acceleration, are applied at non-support nodes, can be made in a similar fashion.

(f) *Lumped mass systems.* If the frame is approximated by a system of masses connected by the massless members, then $\lambda_{jk} = 0$ under the assumption that ${}_jc_k = 0$ ($j = 1, 2, \ldots B$; $k = 1, 2, 3, 4$). In such a case, it can be shown that equations (49) and (50) reduce to

$$\tau = \bar{K}Qu \qquad p + Zu = Q'\tau \tag{57}$$

where $\bar{B}_j$ is used for B_j in equations (45) and (46) and

$$\bar{K} = \begin{bmatrix} \bar{K}_j \end{bmatrix} \qquad j = 1, 2, \ldots B \tag{58*}$$

$$\bar{K}_j = K_{oj} + i\omega C_{oj} \tag{59*}$$

with K_{oj} and C_{oj} being the direct stiffness matrix and damping matrix, respectively, of branch j.

Hence, it follows from equation (57) that

$$\begin{aligned} u &= [Q'\bar{K}Q - Z]^{-1}p \\ \tau &= \bar{K}Q[Q'\bar{K}Q - Z]^{-1}p \end{aligned} \tag{60}$$

(g) *Static analysis.* The same pair of equations as in equation (60) are valid for the static analysis of frames [8, 9] for which $\omega = 0$ and $Z = 0$.

(h) *Frequency response function matrix.* The frequency response function matrices of the nodal displacement and the branch force are obtained respectively from equation (51) and equation (52) by replacing p by a $6N \times 6N$ identity matrix. If, however, lumped mass approximate systems are used, equation (60) is to be used for this purpose.

When the excitation is given at certain nodes (excitation nodes) in the form of displacement or acceleration, the frequency response function matrices of the nodal displacement and the branch force are obtained respectively from equation (55) and equation (56) by replacing u_e by the $6N' \times 6N'$ identity matrix where N' is the number of excitation nodes.

(2) *Trusses*

As pointed out previously, only the first three components of T_j, ${}_FU_j$, ${}_IU_j$, ${}_JP$ and ${}_JU$ are needed for the solution of the truss problem. Although ${}_FU_j$ and ${}_IU_j$ actually have six components, the last three components will be eliminated from the evident condition that hinges cannot resist couples. This condition also eliminates the torsional vibration from formulation. The equations of motion and the end branch force–displacement equations for the other three modes of vibration are the same as those associated with frames.

(a) *Branch vibrations.* First consider the flexural vibration in the x_j–y_j plane. The steady-state solution is given in equation (18), and the boundary conditions remain the same as in equations (20) to (27) except for equations (21) and (23). These two equations should be replaced by

$$\begin{aligned} u''_{j2}(0) &= 0 \\ u''_{j2}(l_j) &= 0 \end{aligned} \tag{61}$$

Then, the transfer equation and the end branch force–displacement relation are obtained as follows.

$${}_F\tau_{j2} = {}_jf\ {}_I\tau_{j2} + {}_j\bar{f}\ {}_Iu_{j2} \tag{62*}$$

$${}_I\tau_{j2} = K_{j2}(b_{j1}\ {}_Iu_{j2} + b_{j2}\ {}_Fu_{j2}) \tag{63*}$$

In a similar procedure, the transfer equation and the end branch force–displacement equation for the flexural vibration in x_j–z_j plane can be obtained, while those for the extensional vibration are identical to those associated with frames and are given in equations (30a) and (31a).

$$ {}_F\tau_{j3} = {}_jg\ {}_I\tau_{j3} + {}_j\bar{g}\ {}_Iu_{j3} \tag{64*} $$

$$ {}_I\tau_{j3} = K_{j3}(\bar{b}_{j1}\ {}_Iu_{j3} + \bar{b}_{j2}\ {}_Fu_{j3}). \tag{65*} $$

Hence, the transfer equations, equations (30a), (62) and (64), and the end branch force–displacement equations, equations (31a), (63) and (65) can be written in the matrix form as follows.

$$ {}_F\tau_j = \mathbf{B}_j\ {}_I\tau_j + \mathbf{D}_j\Lambda_j\ {}_Iu_j^G \tag{66} $$

$$ {}_I\tau_j = K_j(\mathbf{F}_j\Lambda_j\ {}_Iu_j^G + \mathbf{W}_j\Lambda_j\ {}_Fu_j^G) \tag{67} $$

where $\mathbf{B}_j$, $\mathbf{D}_j$, K_j, $\mathbf{F}_j$ and $\mathbf{W}_j$ are all diagonal matrices with diagonal elements as follows

$$ \left[\diagdown\mathbf{B}_j\right] : \cos\lambda_{j1}, \quad {}_jf \quad \text{and} \quad {}_jg \tag{68} $$

$$ \left[\diagdown\mathbf{D}_j\right] : (-m_j\omega^2 + i\omega\ {}_jc_1)l_j \sin\lambda_{j1}/\lambda_{j1}, \quad {}_j\bar{f} \quad \text{and} \quad {}_j\bar{g} \tag{69} $$

$$ \left[\diagdown K_j\right] : K_{j1}, \quad K_{j2} \quad \text{and} \quad K_{j3} \tag{70} $$

$$ \left[\diagdown\mathbf{F}_j\right] : -\lambda_{j1}\cos\lambda_{j1}/\sin\lambda_{j1}, \quad b_{j1} \quad \text{and} \quad \bar{b}_{j1} \tag{71} $$

$$ \left[\diagdown\mathbf{W}_j\right] : \lambda_{j1}/\sin\lambda_{j1}, \quad b_{j2} \quad \text{and} \quad \bar{b}_{j2}. \tag{72} $$

(b) *Nodal vibrations*. The equation of motion (for translation only) of node J can be written in the same form as equation (35).

$$ \sum_j {}_IT_j^G + \sum_k {}_FT_k^G + {}_JP = {}_JM\ {}_J\ddot{U} + {}_J\bar{c}\ {}_J\dot{U} \tag{73} $$

where the index j refers to those branches positively incident on node J whereas k to those negatively incident on node J and

$$ {}_JM = \left[\diagdown{}_JM_j\right], \quad {}_J\bar{c} = \left[\diagdown{}_J\bar{c}_j\right], \quad j = 1, 2, 3 \tag{74} $$

with ${}_JM_1 = {}_JM_2 = {}_JM_3$ being the mass at node J, and ${}_J\bar{c}_1$, ${}_J\bar{c}_2$ and ${}_J\bar{c}_3$ (possibly, ${}_J\bar{c}_1 = {}_J\bar{c}_2 = {}_J\bar{c}_3$) being the coefficients of linear viscous damping associated with the translation of mass in ξ, η and ζ directions.

For the steady state solution of the form equation (14), equation (73) yields, with the aid of equation (66),

$$ {}_Jp + {}_JZ\ {}_Ju - \sum_k \Lambda_k'\mathbf{D}_k\Lambda_k\ {}_Iu_k^G = \sum_k \Lambda_k'\mathbf{B}_k\ {}_I\tau_k - \sum_j \Lambda_j'\ {}_I\tau_j \tag{75} $$

where

$$ {}_jZ = \left[{}_jM_j\omega^2 - i\omega\, {}_j\bar{c}_j \right] \qquad j = 1, 2, 3. \tag{76} $$

(c) *System vibrations.* Define system matrices K and Z in the same way as in the frame analysis, that is, in the form of equations (41) and (42). However, the elements of these matrices are given by equations (70) and (76).

The modified branch–node incidence matrices $\bar{Q}$ and Q are defined as

$$ \begin{aligned} \bar{Q} &= [\bar{q}_{jJ}] \\ Q &= [q_{jJ}] \end{aligned} \tag{77} $$

with

$$ \bar{q}_{jJ} = \begin{cases} 0 & \text{if } a_{jJ} = 0 \\ \mathbf{F}_j\Lambda_j & \text{if } a_{jJ} = 1 \\ \mathbf{W}_j\Lambda_j & \text{if } a_{jJ} = -1 \end{cases} $$

and (78)

$$ q_{jJ} = \begin{cases} 0 & \text{if } a_{jJ} = 0 \\ -\Lambda_j & \text{if } a_{jJ} = 1 \\ \mathbf{B}'_j\Lambda_j & \text{if } a_{jJ} = -1 \end{cases} $$

where $[a_{jJ}]$ is the branch node incidence matrix.

The modified node–node incidence matrix Y is defined as

$$ Y = [y_{IJ}] \tag{79} $$

with

$$ y_{IJ} = \begin{cases} -\Lambda'_k\mathbf{D}_k\Lambda_k & \text{if } e_{IJ} = 1 \text{ and } k \text{ denotes the branch connecting nodes } I \text{ and } J \\ 0 & \text{Otherwise} \end{cases} \tag{80} $$

where $[e_{IJ}]$ is the node–node incidence matrix.

With the aid of these matrices, the branch force–nodal displacement relationship of the system

$$ \tau = K\bar{Q}u \tag{81} $$

follows from equation (67), whereas the equations of motion of non-datum nodes are obtained from equation (75) as

$$ p + (Z + Y)u = Q'\tau \tag{82} $$

Hence, it follows from equations (81) and (82) that

$$ \begin{aligned} u &= [Q'K\bar{Q} - (Z+Y)]^{-1}p \\ \tau &= K\bar{Q}[Q'K\bar{Q} - (Z+Y)]^{-1}p \end{aligned} \tag{83} $$

It should be mentioned that although the same notations, K, Z, Q, $\bar{Q}$ and Y are used for trusses as for frames, their definitions are different.

(d) *Releases and excitations at supports.* Only a slight modification [7] of the preceding formulation makes it possible to consider a roller support. Also, if the excitation is applied at the supports, the formulation has to be modified in the same way as discussed in Section 3 (1) (e).

(e) *Lumped mass systems.* If the dynamic behavior of a truss is approximated by that of a system of masses arranged at non-datum nodes, the analysis is considerably simplified since in this case, only the axial forces exist in the massless branches.

Therefore, redefine the branch quality τ_j as

$$\tau_j = {}_I\tau_j = {}_I\tau_{j1} = {}_F\tau_j = {}_F\tau_{j1} \tag{84}$$

and the various branch matrices are reduced to

$$\begin{aligned} K_j &= A_j E_j (1 + i\omega c_{j1})/l_j \\ \mathbf{F}_j &= -1, \qquad \mathbf{W}_j = +1 \\ \mathbf{B}_j &= +1, \qquad \mathbf{D}_j = 0 \\ \Lambda_j &= [\cos(x_j, \xi) \cos(x_j, \eta) \cos(x_j, \zeta)] \end{aligned} \tag{85}$$

The two modified branch–node incidence matrices are then identical and the modified node–node incidence matrix is zero; $Q = \bar{Q}$ and $Y = 0$. Therefore, it follows that the equation of motion and the branch force–nodal displacement relationship can be written from equations (81) and (82) as

$$p + Zu = Q'\tau \qquad \tau = KQu \tag{86}$$

with τ being a $(B \times 1)$ column matrix.

(f) *Static analysis.* The same pair of equations as in equation (86) are valid for static analysis of trusses [10] for which $z = 0$.

(g) *Frequency response function matrices.* The frequency response function matrices of the nodal displacement and the branch force are obtained from equation (83) (from equation (86) if lumped mass approximation is used) by replacing p by the $3N \times 3N$ identity matrix.

4. IMPULSE RESPONSE FUNCTION MATRICES

The impulse response function for a complex, distributed mass structure can be evaluated numerically either by the numerical Fourier inversion from the frequency response function obtained previously [equation (3)] or by the Laplace transform technique, while the impulse response function of a lumped mass structure can be evaluated by finding the eigen values and eigen vectors of a certain pertinent matrix [11].

(1) *Lumped mass systems*

For the steady state solution obtained previously, the equations relating τ_j, u_j, ${}_jp$, ${}_ju$ etc. are nothing but the Fourier transform of the corresponding equations relating T_j, U_j, ${}_jP$, ${}_jU$ etc.

It can be shown that the Fourier inversion of equations (57) (the frame problem) are

$$T = K_o QU + C_o Q\dot{U} \tag{87}$$

$$M\ddot{U} + C\dot{U} + Q'T = P \tag{88}$$

where

$$K_o = \left[K_{oj} \right], \qquad C_o = \left[C_{oj} \right], \qquad j = 1, 2, \ldots B \tag{89}$$

$$C = \left[{}_J\bar{c} \right], \qquad M = \left[{}_J M \right], \qquad J = 1, 2, \ldots N \tag{90}$$

with K_{oj}, C_{oj}, ${}_J\bar{c}$ and ${}_JM$ being given in equations (59), (36) and (37), respectively.

It follows from equations (87) and (88) that

$$M\ddot{U} + (C + Q'C_oQ)\dot{U} + Q'K_oQU = P \tag{91}$$

By introducing a new variable [11]

$$X = [\dot{U}' \,\vdots\, U']' \tag{92}$$

equations (91) and (92) can be written as

$$\dot{X} + GX = \begin{bmatrix} M^{-1} \\ \hline 0 \end{bmatrix} P \tag{93}$$

with

$$G = \left[\begin{array}{c|c} M^{-1}\bar{C} & M^{-1}\bar{K} \\ \hline -I_N & 0 \end{array} \right] \tag{94}$$

where I_N is the $6N \times 6N$ identity matrix, $\bar{C} = C + Q'C_oQ$ and $\bar{K} = Q'K_oQ$.

The impulse response function matrix is then obtained as

$$h(t) = \begin{cases} [0 \,\vdots\, I_N] E D(t) E^{-1} \begin{bmatrix} M^{-1} \\ \hline 0 \end{bmatrix} & t \geqq 0 \\ 0 & t < 0 \end{cases} \tag{95}$$

where $D(t) = \left[e^{-\lambda_j t} \right]$ with λ_j being the jth eigenvalue of G and E is the modal matrix of G.

Equation (91) and therefore equation (95) are also valid for a lumped mass truss if appropriate matrices defined in section 3 (2) (e) are used.

If the excitation is applied at supports in the form of displacement or acceleration partitions of U and Q can be made in the same way as in equations (53) and (54) and the impulse response function matrix can be obtained in a similar fashion.

(2) *Distributed mass systems*

(a) *Laplace transform of basic equations and their inversions.* Let γ be the Laplace transform parameter. It can be shown that, under zero initial condition, the solutions T and U in the γ plane can be obtained from the steady-state solutions equations (51), (52) and (83) if the frequency ω appearing in these equations is replaced by $-i\gamma$. Hence, a method of numerical Laplace inversion, proposed by Wing [12], which is a modification of that by Weeks [13] making use of the Cooley–Tukey algorithm [14], can be employed.

(b) *Numerical Fourier inversion.* It is stated in equation (3) that the impulse response function is nothing but the Fourier inversion of the corresponding frequency response function. A method of the numerical Fourier inversion developed by the present authors [7], based on complex Fourier series technique and the Cooley–Tukey algorithm [14] is conveniently used.

The experience indicates that these two approaches require approximately the same amount of computational work.

5. NUMERICAL EXAMPLES

It is important to recall the fact that the impulse response function of a complex, distributed mass structure can only be evaluated "numerically" either as the Fourier inversion of the frequency response function or through the Laplace inversion as described in Section 4 (2) (a).

It is found that when the covariance function as well as the variance function of the response of such a structure is to be computed, as in the following examples, the frequency domain analysis is much more practical than the time domain analysis. This is because, for the time domain analysis, the impulse response function, which can only be estimated on an extra step involving either Fourier inversion or Laplace inversion as described above, is required and, moreover the double convolution integral [equation (9)] has to be carried out whereas, for the frequency domain analysis, it is necessary only to evaluate equation (10) using the numerical method of double Fourier inversion [7] involving the frequency response function and the generalized spectral density of the excitation only.

This argument is obviously based on the assumption that the covariance function and the generalized spectral density of the excitation can be estimated with approximately equal ease. At the present time, no method exists to estimate, with any degree of confidence, the covariance function of a nonstationary process on a single sample function (or even on a few number of sample functions), and no physical significance of practical use is known for the generalized spectral density. Therefore, usually, one has to be satisfied with the covariance function and the associated generalized spectral density of a time domain "model" of the excitation process exhibiting reasonably well a general trend of variance function observed in the sample function(s) and reproducing a covariance function within a time interval with an apparent (local) stationarity which is in a reasonable agreement with covariance values computed from the sample function(s) assuming the ergodicity. Since it is not difficult to construct such a time domain model with an analytically well-defined generalized spectral density [15], the assumption that the covariance function and the generalized spectral density can be estimated with more or less equal ease, seems justified. In fact, a generalized spectral density derived from such a time domain model is used in the frequency domain analysis in Example 1.

(1) *Example* 1

A two-story plane frame is idealized by (i) a distributed mass system (Structure I) as shown in Fig. 3 and by (ii) a lumped mass shear beam structure (Structure II) as shown in Fig. 4. The mechanical properties of Structures I and II are listed in Tables 1 and 2 respectively. The masses lumped at joint A and B of Structure II are, respectively, equal to the total mass of the 1st and the 2nd floor. The external damping associated with branch and nodal vibrations is assumed to be zero. The interior damping associated with the branch vibrations is considered in the form of linear viscous damping. The frequency

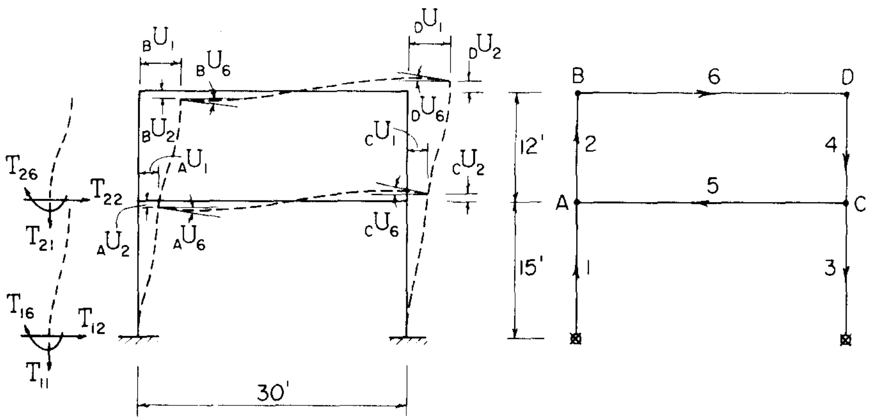

FIG. 3. Structure I and its graph.

response functions of both Structure I and II to the acceleration excitation $e^{i\omega t}$ applied at the foundation of the structures are obtained. The absolute values of the frequency response functions are plotted in Figs. 5 and 6.

TABLE 1. MECHANICAL PROPERTIES OF STRUCTURE I

Branch	Mass per unit length (lb. sec^2/in^2)	Area (in^2)	Moment of inertia (in^4)	Retardation time (sec)
1	0·01	13·24	248·6	0·008
3	0·01	13·24	248·6	0·008
2	0·005	6·19	106·3	0·013
4	0·005	6·19	106·3	0·013
5	0·55	24·7	2364·3	0·008
6	0·41833	18·23	1326·8	0·013

TABLE 2. MECHANICAL PROPERTIES OF STRUCTURE II

Node	Lumped mass (lb. sec^2/in)
A	198
B	150·6

Branch	Area (in^2)	Moment of inertia (in^4)	Retardation time (sec)
1	26·48	497·2	0·008
2	12·38	212·6	0·013

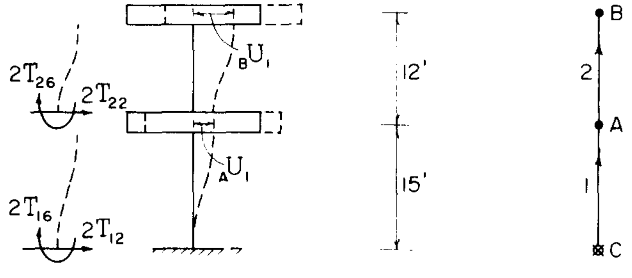

FIG. 4. Structure II and its graph.

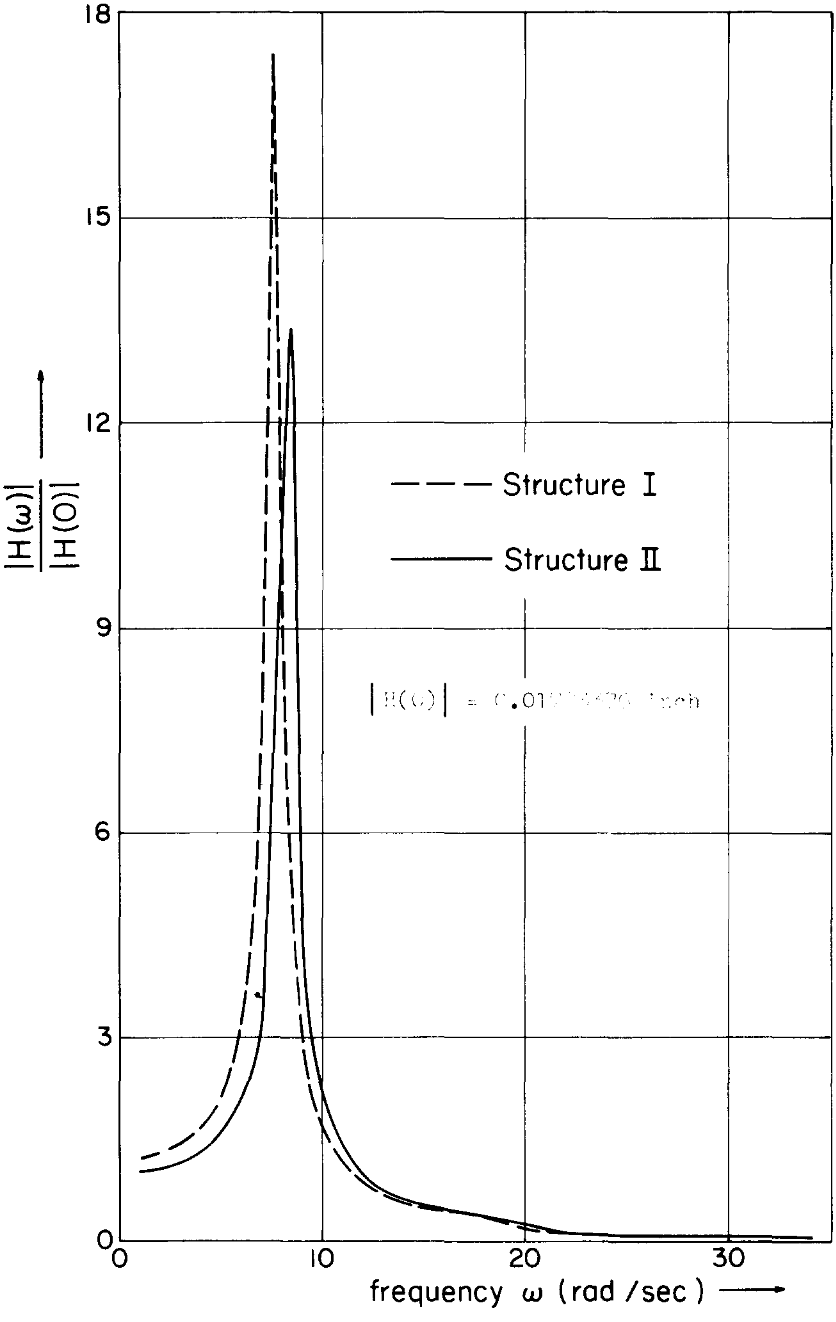

FIG. 5. Frequency response function of 2nd floor relative displacement ${}_BU_I$.

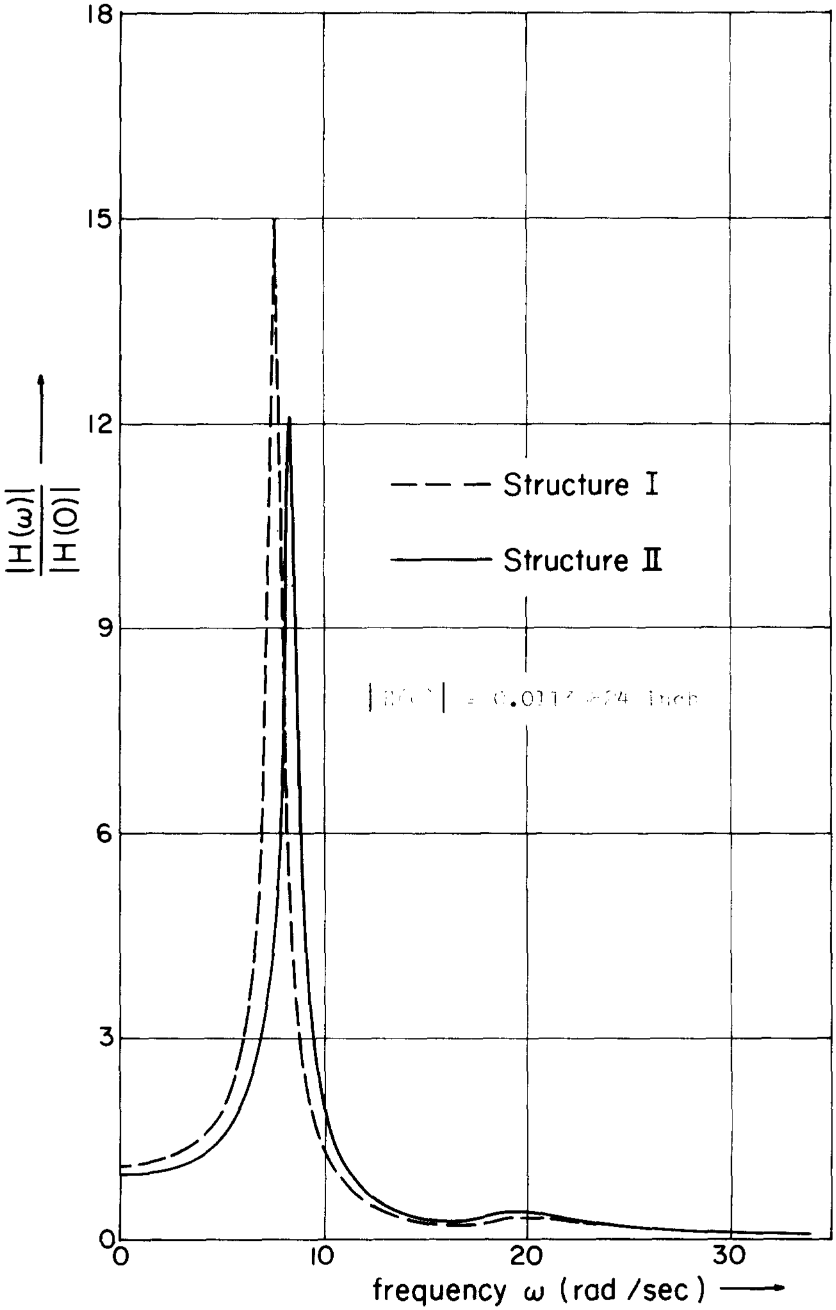

FIG. 6. Frequency response function of 1st floor relative displacement $_AU_I$.

The impulse response functions of both structures to the acceleration $\delta(t)$ applied at the foundation are plotted in Figs. 7 and 8. Both the Laplace transform and the Fourier inversion techniques are employed for Structure I. Practically no difference in numerical results is observed between these two techniques employed.

The variance functions of the response of both structures to an artificial earthquake excitation are plotted in Figs. 9 and 10. The earthquake acceleration is simulated by passing a nonstationary shot noise through an appropriate filter as discussed in [15]. The frequency domain analysis [equation (10)] is used for Structure I with the aid of numerical double Fourier inversion technique developed in [7], whereas both the time domain analysis [equation (9)] and the frequency domain analysis [equation (10)] are employed for Structure II. In the latter case, it is noted that both analyses yield practically the same numerical result.

It is observed from these figures that the fundamental frequency of Structure I is lower than that of Structure II and at these fundamental frequencies the relative displacements of ${}_AU_1$ and ${}_BU_1$ of Structure I are larger than those of Structure II. This is due to the shear beam idealization of the building implying that Structure II is more rigid than Structure I and has therefore a higher resonant frequency and smaller horizontal displacements. The shapes of the frequency response functions of both structures are almost identical. This is because, in this particular example, the masses of floor systems are large in comparison with those of columns so that the fundamental mode dominates the dynamic behavior of both structures when the external excitation is applied at the foundation. Similar conclusion can be drawn from the observation of impulse response functions.

To get some ideas on the degree of the internal damping, it is noted that the retardation times assumed in Tables 1 and 2 produce the damping ratio of the order of 5 per cent for the

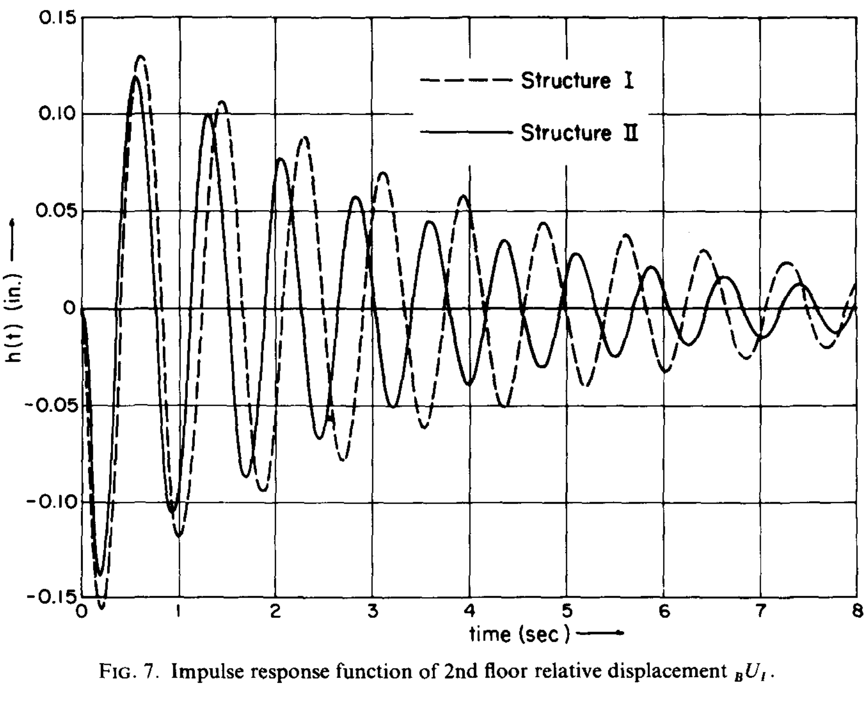

FIG. 7. Impulse response function of 2nd floor relative displacement ${}_BU_l$.

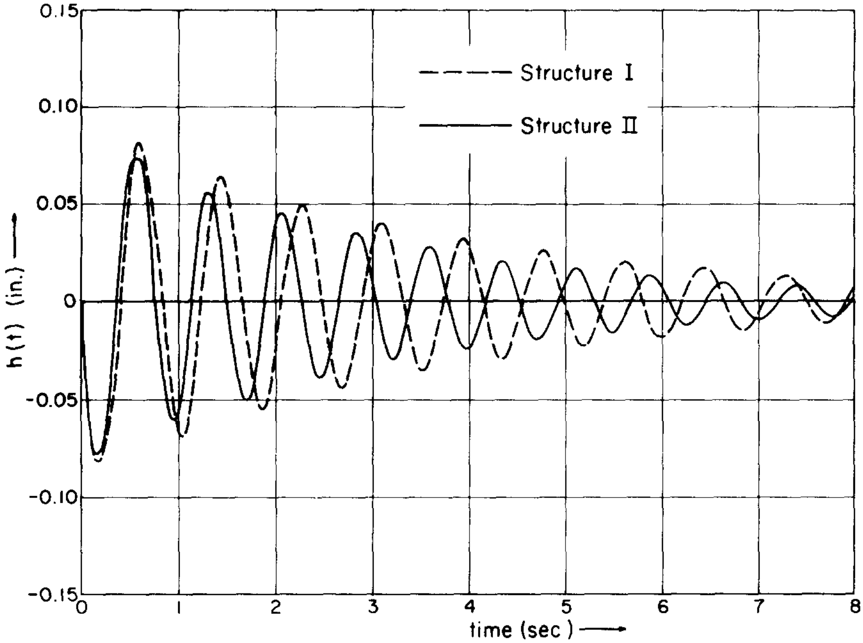

FIG. 8. Impulse response function of 1st floor relative displacement ${}_AU_l$.

first mode of vibration when the classical modal analysis of the lumped mass system is considered in approximation.

(2) *Example* 2

A plane truss is treated as (i) a continuous mass structure (Structure III) and as (ii) a lumped mass structure (Structure IV) as shown in Fig. 11. The mechanical properties of both structures are listed in Tables 3 and 4. The external damping associated with branch

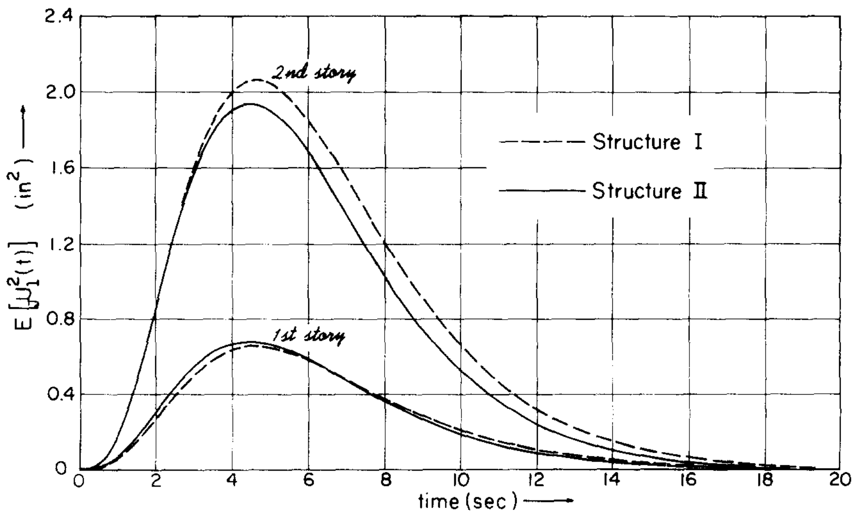

FIG. 9. Variance function $E[{}_JU_1^2(t)]$ of floor relative displacement ${}_JU_1(t)$, $J = A, B$.

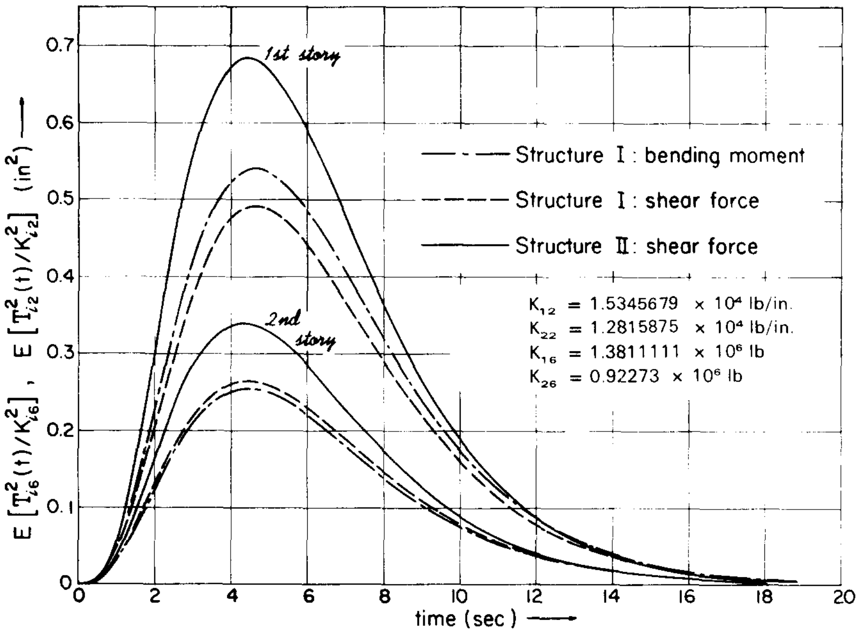

FIG. 10. Variance function $E[T_{i2}^2(t)/K_{i2}^2]$ of story shear force T_{i2} and $E[T_{i6}^2(t)/K_{i6}^2]$ of story bending moment T_{i6}, where K_{i2} and K_{i6} are the shear stiffness and bending stiffness of ith column of Structure I.

and nodal vibrations is assumed to be zero. The interior damping associated with branch vibrations is considered as (i) linear viscous damping and (ii) complex damping. The retardation time and complex damping coefficient for each member are assumed to be 0·00015 sec and 0·01, respectively, for both structures. The lumped mass at each joint of

TABLE 3. MECHANICAL PROPERTIES OF STRUCTURE III

Branch	Mass per unit length (lb. sec²/in²)	Area (in²)	Moment of inertia (in⁴)
1	0·025	30·0	300·0
2	0·025	30·0	300·0
3	0·025	30·0	300·0
4	0·025	30·0	300·0
5	0·0151	20·0	250·0
6	0·0825	30·0	300·0
7	0·0825	30·0	300·0

TABLE 4. MECHANICAL PROPERTIES OF STRUCTURE IV

Node	Lumped mass (lb. sec²/in)
A	5·637
B	5·637
C	23·625
D	11·8125

Note: the area and the moment of inertia of each branch are given in Table 3. The mass per unit length of each branch is zero.

Structure IV is equal to one half of the sum of the masses of all the members that are connected to the corresponding joint in Structure III. The frequency response functions of both structures to a concentrated force $e^{i\omega t}$ acting at node 3 in the vertical direction are obtained. The absolute values of the frequency response functions are plotted in Figs. 12 and 13.

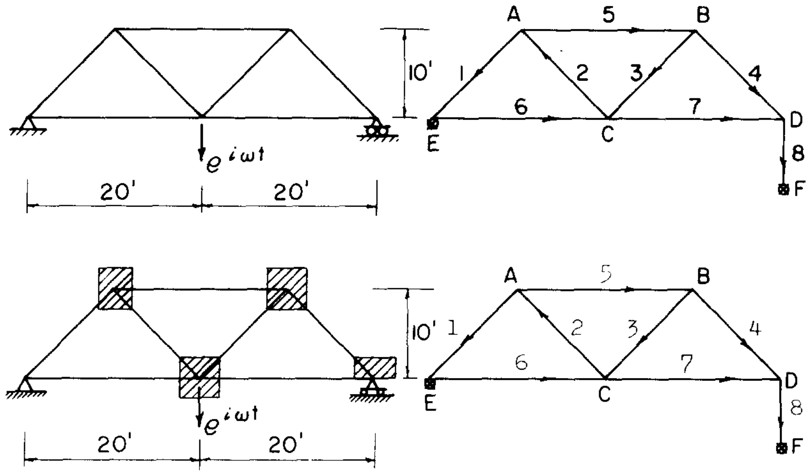

FIG. 11. (a) Structure III and its graph. (b) Structure IV and its graph.

A considerable difference is observed between the frequency response function of Structure III and that of Structure IV. This indicates that the idealization of a truss (without a heavy floor system) by a lumped mass system as considered here is not reasonable in the dynamic analysis, except that the first and the second natural frequencies may be estimated in approximation from the lumped mass system as suggested in Figs. 12 and 13.

The frequency response functions of the structure with the viscous damping and those with complex damping are almost identical in the vicinity of $\omega = 66{\cdot}7$ rad/sec as it should be. Since the frequency response function with complex damping is obtained from that with linear viscous damping of the Kelvin type by replacing $ic_{jk}\omega$ by $i\alpha_{jk}$, these response functions take an identical value at $\omega = \alpha_{jk}/c_{jk}$, in the present example, $\omega = 66{\cdot}7$ rad/sec. For the frequencies greater than 66·7 rad/sec, the response of the structure with viscous damping is smaller than that of the structure with the complex damping as can be seen from the figures. This is because the viscous damping is larger than the complex damping in this domain of frequency.

It is noted that, in this example, the roller support is considered as non-datum node by adding a massless fictitious branch 8, with a very large extensional stiffness and zero damping (interior and exterior) so that the formulation can be applied without the modification mentioned in Section 3(2)(d).

As in Example 1, if the classical modal analysis is considered for the lumped mass system in approximation for the purpose of assessing the internal damping associated with the retardation time of 0·00015 sec in more familiar terms, it produces a damping ratio in the neighborhood of 1·5 per cent for the first mode.

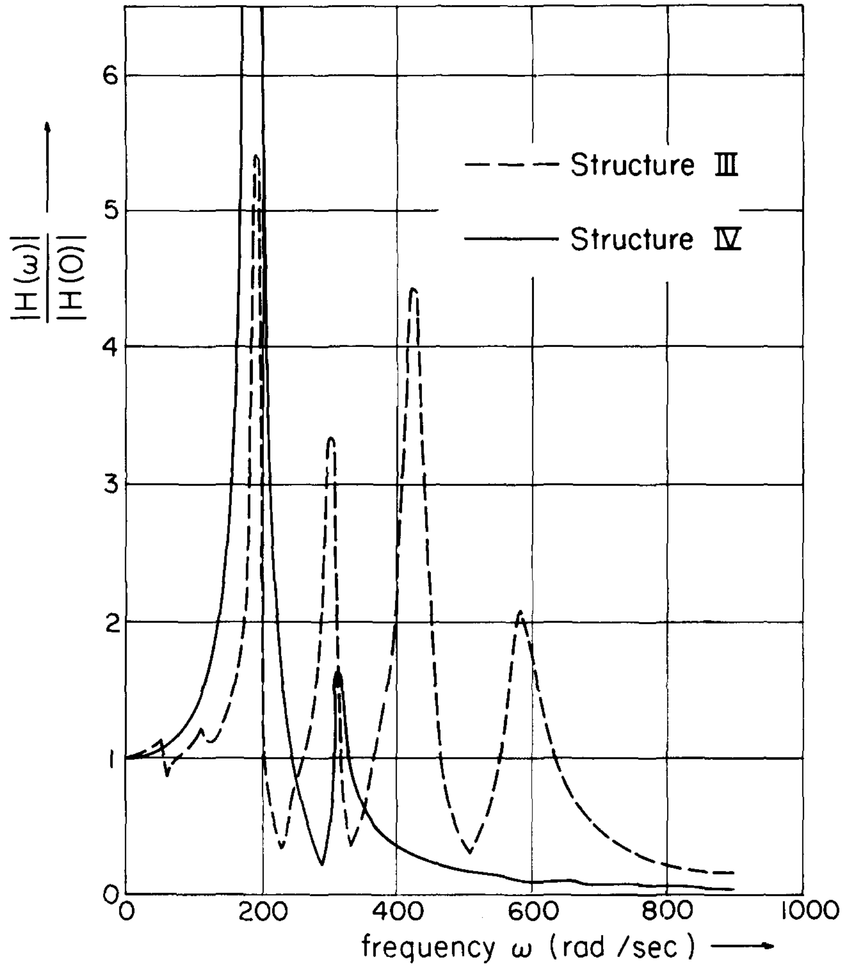

FIG. 12. Frequency response function of vertical displacement at joint C (viscous damping). $|H(0)| = 0{\cdot}91046 \times 10^{-3}$ in.

6. CONCLUSION

Systematical methods are presented for the determination of the frequency and the impulse response functions of linear structures that are considered either as lumped mass systems or as distributed mass systems. Network concept and transfer function technique are employed throughout the formulation so that the geometric configuration of the structures can be taken into account in a general fashion. This permits a convenient use of a digital computer for the numerical work involved in the analysis. It is shown that the general formulation of the dynamic problem is degenerated into that of static analysis when the frequency of excitation is set to be zero.

Use of a numerical method of the single and the double inversion of the Fourier transform using complex Fourier series technique is emphasized since this technique makes the frequency domain analysis of the nonstationary random vibration tractable. The methods

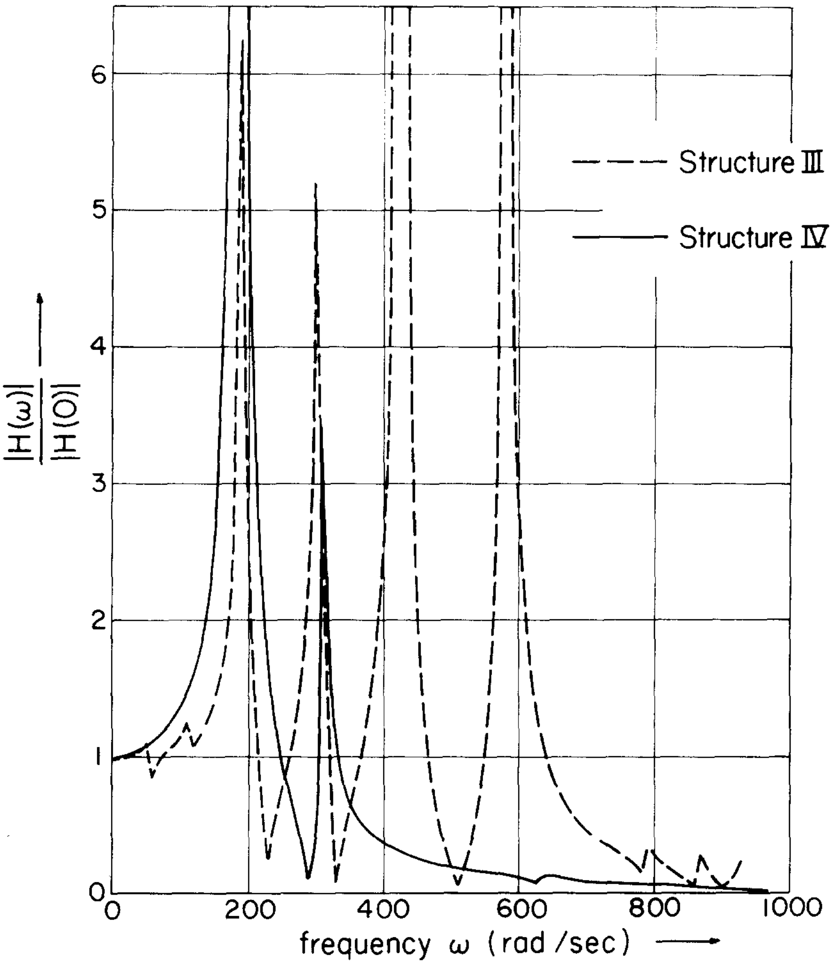

FIG. 13. Frequency response function of vertical displacement at joint C (complex damping), $|H(0)| = 0{\cdot}91046 \times 10^{-3}$ in.

presented in this study constitute a unified systematic approach for the evaluation of the statistical quantities such as the mean value function and the cross covariance function of the response of complex linear structures.

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APPENDIX—QUANTITIES AND SYMBOLS GIVEN IN TEXT

(A) *Frames*

$$\begin{bmatrix} {}_jf_{11} & {}_jf_{12} \\ {}_jf_{21} & {}_jf_{22} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cosh\lambda_{j2} + \cos\lambda_{j2} & -\frac{\lambda_{j2}}{l_j}(\sinh\lambda_{j2} - \sin\lambda_{j2}) \\ -\frac{l_j}{\lambda_{j2}}(\sinh\lambda_{j2} + \sin\lambda_{j2}) & \cosh\lambda_{j2} + \cos\lambda_{j2} \end{bmatrix}$$

$$\begin{bmatrix} {}_j\bar{f}_{11} & {}_j\bar{f}_{12} \\ {}_j\bar{f}_{21} & {}_j\bar{f}_{22} \end{bmatrix} = \begin{bmatrix} -\frac{\sinh\lambda_{j2} + \sin\lambda_{j2}}{2\lambda_{j2}} l_j & -\frac{\cosh\lambda_{j2} - \cos\lambda_{j2}}{2\lambda_{j2}^2} l_j^2 \\ \frac{\cosh\lambda_{j2} - \cos\lambda_{j2}}{2\lambda_{j2}^2} l_j^2 & \frac{\sinh\lambda_{j2} - \sin\lambda_{j2}}{2\lambda_{j2}^3} l_j^3 \end{bmatrix} . (m_j\omega^2 - i\omega_j c_2)$$

$$\begin{bmatrix} {}_jb_{11} & {}_jb_{12} & {}_jb_{13} & {}_jb_{14} \\ {}_jb_{21} & {}_jb_{22} & {}_jb_{23} & {}_jb_{24} \end{bmatrix} = \begin{bmatrix} \lambda_{j2}^3 & 0 & -\lambda_{j2}^3 & 0 \\ 0 & -\lambda_{j2}^2 & 0 & \lambda_{j2}^2 \end{bmatrix} \begin{bmatrix} \\ A_j \\ \\ \end{bmatrix}^{-1}$$

$$
\begin{bmatrix} \\ A_j \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \dfrac{\lambda_{j2}}{l_j} & 0 & \dfrac{\lambda_{j2}}{l_j} & 0 \\ \sin\lambda_{j2} & \cos\lambda_{j2} & \sinh\lambda_{j2} & \cosh\lambda_{j2} \\ \dfrac{\lambda_{j2}}{l_j}\cos\lambda_{j2} & -\dfrac{\lambda_{j2}}{l_j}\sin\lambda_{j2} & \dfrac{\lambda_{j2}}{l_j}\cosh\lambda_{j2} & \dfrac{\lambda_{j2}}{l_j}\sinh\lambda_{j2} \end{bmatrix}
$$

$$
\begin{aligned}
K_{j1} &= A_jE_j(1+i\omega c_{j1})/l_j, \qquad K_{j2} = E_jI_{jz}(1+i\omega c_{j2})/l_j^3 \\
K_{j3} &= E_jI_{jy}(1+i\omega c_{j3})/l_j^3, \qquad K_{j4} = G_jJ_j(1+i\omega c_{j4})/l_j \\
K_{j5} &= E_jI_{jy}(1+i\omega c_{j3})/l_j^2, \qquad K_{j6} = E_jI_{jz}(1+i\omega c_{j2})/l_j^2 \\
\lambda_{j1}^2 &= (m_j\omega^2 - i\omega_j c_1)l_j^2/[A_jE_j(1+i\omega c_{j1})] \\
\lambda_{j3}^4 &= (m_j\omega^2 - i\omega\, {}_jc_3)l_j^4/[E_jI_{jy}(1+i\omega c_{j3})] \\
\lambda_{j4}^2 &= (m_jr_j^2\omega^2 - i\omega\, {}_jc_4)l_j^2/[G_jJ_j(1+i\omega c_{j4})]
\end{aligned}
$$

$$
\begin{bmatrix} {}_jg_{11} & {}_jg_{12} \\ {}_jg_{21} & {}_jg_{22} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cosh\lambda_{j3}+\cos\lambda_{j3} & \dfrac{\lambda_{j3}}{l_j}(\sinh\lambda_{j3}-\sin\lambda_{j3}) \\ \dfrac{l_j}{\lambda_{j3}}(\sinh\lambda_{j3}+\sin\lambda_{j3}) & \cosh\lambda_{j3}+\cos\lambda_{j3} \end{bmatrix}
$$

$$
\begin{bmatrix} {}_j\bar{g}_{11} & {}_j\bar{g}_{12} \\ {}_j\bar{g}_{21} & {}_j\bar{g}_{22} \end{bmatrix} = \begin{bmatrix} -\dfrac{\sinh\lambda_{j3}+\sin\lambda_{j3}}{\lambda_{j3}}l_j & \dfrac{\cosh\lambda_{j3}-\cos\lambda_{j3}}{\lambda_{j3}^2}l_j^2 \\ -\dfrac{\cosh\lambda_{j3}-\cos\lambda_{j3}}{\lambda_{j3}^2}l_j^2 & \dfrac{\sinh\lambda_{j3}-\sin\lambda_{j3}}{\lambda_{j3}^3}l_j^3 \end{bmatrix} \cdot \frac{m_j\omega^2 - i\omega\, {}_jc_3}{2}
$$

$$
\begin{bmatrix} {}_j\bar{b}_{11} & {}_j\bar{b}_{12} & {}_j\bar{b}_{13} & {}_j\bar{b}_{14} \\ {}_j\bar{b}_{21} & {}_j\bar{b}_{22} & {}_j\bar{b}_{23} & {}_j\bar{b}_{24} \end{bmatrix} = \begin{bmatrix} \lambda_{j3}^3 & 0 & -\lambda_{j3}^3 & 0 \\ 0 & \lambda_{j3}^2 & 0 & -\lambda_{j3}^2 \end{bmatrix} \begin{bmatrix} \\ \bar{A}_j \\ \\ \end{bmatrix}^{-1}
$$

$$\left[\bar{A}_j\right] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\dfrac{\lambda_{j3}}{l_j} & 0 & -\dfrac{\lambda_{j3}}{l_j} & 0 \\ \sin\lambda_{j3} & \cos\lambda_{j3} & \sinh\lambda_{j3} & \cosh\lambda_{j3} \\ -\dfrac{\lambda_{j3}}{l_j}\cos\lambda_{j3} & \dfrac{\lambda_{j3}}{l_j}\sin\lambda_{j3} & -\dfrac{\lambda_{j3}}{l_j}\cosh\lambda_{j3} & -\dfrac{\lambda_{j3}}{l_j}\sinh\lambda_{j3} \end{bmatrix}$$

$$B_j = \begin{bmatrix} \cos\lambda_{j1} & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_jf_{11} & 0 & 0 & 0 & {}_jf_{12} \\ 0 & 0 & {}_jg_{11} & 0 & {}_jg_{12} & 0 \\ 0 & 0 & 0 & \cos\lambda_{j4} & 0 & 0 \\ 0 & 0 & {}_jg_{21} & 0 & {}_jg_{22} & 0 \\ 0 & {}_jf_{21} & 0 & 0 & 0 & {}_jf_{22} \end{bmatrix}$$

$$D_j = \begin{bmatrix} {}_jd_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_j\bar{f}_{11} & 0 & 0 & 0 & {}_j\bar{f}_{12} \\ 0 & 0 & {}_j\bar{g}_{11} & 0 & {}_j\bar{g}_{12} & 0 \\ 0 & 0 & 0 & {}_jd_4 & 0 & 0 \\ 0 & 0 & {}_j\bar{g}_{21} & 0 & {}_j\bar{g}_{22} & 0 \\ 0 & {}_j\bar{f}_{21} & 0 & 0 & 0 & {}_j\bar{f}_{22} \end{bmatrix}$$

with

$$ {}_jd_1 = -(m_j\omega^2 - i\omega\,{}_jc_1)l_j\sin\lambda_{j1}/\lambda_{j1}$$
$$ {}_jd_4 = -(m_jr_j^2\omega^2 - i\omega\,{}_jc_4)l_j\sin\lambda_{j4}/\lambda_{j4}$$

$$F_j = \begin{bmatrix} -\dfrac{\lambda_{j1}}{\sin\lambda_{j1}}\cos\lambda_{j1} & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_jb_{11} & 0 & 0 & 0 & {}_jb_{12} \\ 0 & 0 & {}_j\bar{b}_{11} & 0 & {}_j\bar{b}_{12} & 0 \\ 0 & 0 & 0 & -\dfrac{\lambda_{j4}}{\sin\lambda_{j4}}\cos\lambda_{j4} & 0 & 0 \\ 0 & 0 & {}_j\bar{b}_{21} & 0 & {}_j\bar{b}_{22} & 0 \\ 0 & {}_jb_{21} & 0 & 0 & 0 & {}_jb_{22} \end{bmatrix}$$

$$W_j = \begin{bmatrix} \dfrac{\lambda_{j1}}{\sin \lambda_{j1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_jb_{13} & 0 & 0 & 0 & {}_jb_{14} \\ 0 & 0 & {}_j\bar{b}_{13} & 0 & {}_j\bar{b}_{14} & 0 \\ 0 & 0 & 0 & \dfrac{\lambda_{j4}}{\sin \lambda_{j4}} & 0 & 0 \\ 0 & 0 & {}_j\bar{b}_{23} & 0 & {}_j\bar{b}_{24} & 0 \\ 0 & {}_jb_{23} & 0 & 0 & 0 & {}_jb_{24} \end{bmatrix}$$

$$\bar{B}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & l_j & 0 & 1 & 0 \\ 0 & -l_j & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{F}_j = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & -6l_j \\ 0 & 0 & -12 & 0 & 6l_j & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 & -4l_j & 0 \\ 0 & -6 & 0 & 0 & 0 & -4l_j \end{bmatrix}$$

$$\bar{W}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & -6l_j \\ 0 & 0 & 12 & 0 & 6l_j & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & -2l_j & 0 \\ 0 & 6 & 0 & 0 & 0 & -2l_j \end{bmatrix} \qquad \bar{D}_j = [0]$$

$$\bar{K}_j = K_{0j} + i\omega C_{0j} = \begin{bmatrix} \frac{A_jE_j(1+i\omega c_{j1})}{l_j} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12E_jI_{jz}(1+i\omega c_{j2})}{l_j^3} & 0 & 0 & 0 & \frac{6E_jI_{jz}(1+i\omega c_{j2})}{l_j^2} \\ 0 & 0 & \frac{12E_jI_{jy}(1+i\omega c_{j3})}{l_j^3} & 0 & \frac{-6E_jI_{jy}(1+i\omega c_{j3})}{l_j^2} & 0 \\ 0 & 0 & 0 & \frac{G_jJ_j(1+i\omega c_{j4})}{l_j} & 0 & 0 \\ 0 & 0 & \frac{-6E_jI_{jy}(1+i\omega c_{j3})}{l_j^2} & 0 & \frac{4E_jI_{jy}(1+i\omega c_{j3})}{l_j} & 0 \\ 0 & \frac{6E_jI_{jz}(1+i\omega c_{j2})}{l_j^2} & 0 & 0 & 0 & \frac{4E_jI_{jz}(1+i\omega c_{j2})}{l_j} \end{bmatrix}$$

(B) *Trusses*

$$_jf = \frac{\cos\lambda_{j2}\sinh\lambda_{j2} - \sin\lambda_{j2}\cosh\lambda_{j2}}{\sinh\lambda_{j2} - \sin\lambda_{j2}}$$

$$_j\bar{f} = \left[\frac{(\cosh\lambda_{j2} - \cos\lambda_{j2})^2}{2\lambda_{j2}(\sinh\lambda_{j2} - \sin\lambda_{j2})} l_j - \frac{\sinh\lambda_{j2} + \sin\lambda_{j2}}{2\lambda_{j2}} l_j\right](m_j\omega^2 - i\omega\, _jc_2)$$

$$b_{j1} = \frac{\lambda_{j2}^2}{2}\left[\frac{\lambda_{j2}}{\sinh\lambda_{j2}}\cosh\lambda_{j2} - \frac{\lambda_{j2}}{\sin\lambda_{j2}}\cos\lambda_{j2}\right]$$

$$b_{j2} = \frac{\lambda_{j2}^2}{2}\left[\frac{\lambda_{j2}}{\sin\lambda_{j2}} - \frac{\lambda_{j2}}{\sinh\lambda_{j2}}\right]$$

$$K_{j2} = \frac{E_jI_{jz}(1+i\omega c_{j2})}{l_j^3}$$

$$\lambda_{j2}^4 = \frac{m_j\omega^2 - i\omega_jc_2}{E_jI_{jz}(1+i\omega c_{j2})} l_j^4$$

$$_jg = \frac{\cos\lambda_{j3}\sinh\lambda_{j3} - \sin\lambda_{j3}\cosh\lambda_{j3}}{\sinh\lambda_{j3} - \sin\lambda_{j3}}$$

$$_j\bar{g} = \left[\frac{(\cosh\lambda_{j3} - \cos\lambda_{j3})^2}{2\lambda_{j3}(\sinh\lambda_{j3} - \sin\lambda_{j3})} l_j - \frac{\sinh\lambda_{j3} + \sin\lambda_{j3}}{2\lambda_{j3}} l_j\right](m_j\omega^2 - i\omega\, _jc_3)$$

$$\lambda_{j3}^4 = \frac{m_j\omega^2 - i\omega\, _jc_3}{E_jI_{jy}(1+i\omega c_{j3})} l_j^4$$

$$\bar{b}_{j1} = \frac{\lambda_{j3}^2}{2}\left(\frac{\lambda_{j3}}{\sinh\lambda_{j3}}\cosh\lambda_{j3} - \frac{\lambda_{j3}}{\sin\lambda_{j3}}\cos\lambda_{j3}\right)$$

$$\bar{b}_{j2} = \frac{\lambda_{j3}^2}{2}\left(\frac{\lambda_{j3}}{\sin\lambda_{j3}} - \frac{\lambda_{j3}}{\sinh\lambda_{j3}}\right)$$

$$K_{j3} = \frac{E_jI_{jy}(1+i\omega c_{j3})}{l_j^3}$$

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Абстрак—Работа представляет обьединенный метод динамического расчета конструкций, который легко применить к расчетам анализа нестационарной произвольной реакции линейных устойчиых ферм и рам, обладающих двух-или трехмерной конфигурацией. Вообще метод является достаточным для рассматривания конструкции так в виде непрерывной (распределенной) системы масс, как и дискретной (сосредоточенной) системы масс, с какой нибудь формой вязкого демпфирования. Всегда можно использовать функции сил, произвольных во времени и/или пространстве, если рассматривается рама. Однако для случая фермы, силы учитываются только в узлах.

Выводя матрицы функции реакции частоты или матрицы функции реакции импульса, применяется линейная теория графов и способ мранспонированной матрицы при всей формулировке. Конфигурация обсуждаемых конструкций, принята найболее общим способом, дозваляет использовать для численных расчетов быстродействующую цифровую вычислительную машину. Предложенная формулировка заключает, в качестве специального случая, статический метод расчета конструкций.

Разработано некоторые количество численных примеров. Сравниваются динамические характеристики непрерывных систем масс с соответственными системами дискретных масс.